Research on Teaching Strategies of the Dual Simplex Method

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Abstract

The dual simplex method for linear programming problems has always been a key and challenging topic in the teaching of operations research. By developing a thorough understanding of the dual simplex method and aiming to design an accessible teaching approach, this paper systematically designs a teaching strategy for the dual simplex method. The goal is to help students more easily understand its principles and grasp the relationship between the dual simplex method and the simplex method. The model of using a single numerical example throughout the entire teaching process is simpler and more intuitive, allowing for a more systematic introduction to the dual simplex method and enabling students to better comprehend the essence of the algorithm.

Keywords

Simplex method; dual simplex method; principle; teaching.

1. Introduction

Linear programming is an important branch of operations research and serves as a commonly used mathematical model for solving many practical problems in modern management [1-4]. The simplex method is the primary approach for solving linear programming problems[5], while the dual simplex method is another solution technique that can sometimes improve computational efficiency[2,6]. However, the mathematical description of the dual simplex method is abstract and not easy to grasp[7]. In teaching practice, we have observed that students' understanding of the dual simplex method is much weaker than that of the simplex method. This is mainly due to two reasons: first, in most textbooks, the presentation of the matrix form of the simplex method, as well as the relationship between the primal and dual problems, tends to be scattered and lacks coherence; second, the introduction of the dual simplex method is often heavily tied to abstract mathematical symbols and derivations, which makes it difficult for students with weaker mathematical foundations to absorb.

The fundamental idea of improving the teaching of operations research courses lies in reducing emphasis on abstract mathematical derivations and calculations, while making use of more relatable, practical examples [8,9], which help students truly understand the essence of the algorithm. Therefore, based on a thorough analysis of the teaching focus and difficulties of the dual simplex method, and taking into account students' learning progress, this paper systematically organizes the underlying logic of the dual simplex method by employing a numerical example throughout the entire teaching process. The proposed teaching strategy presents the matrix form of the simplex method, the relationship between the solutions of the primal and dual problems, and the principles of the dual simplex method. Through step-by-step explanations with examples, the strategy aims to clarify the principles of the dual simplex method in a systematic, coherent, and accessible manner.

2. Standard Form of the Primal and Dual Problems

Given that many operations research textbooks discuss models in the form of maximization problems, the linear programming model considered in this paper [1] is as follows:

ISSN: 2637-6067 DOI: 10.6918/IJOSSER.202510_8(10).0006

$$\begin{aligned} \max \quad & z = CX \\ s.t. \quad & AX \le b \\ & X \ge 0 \end{aligned} \tag{1}$$

Its standard form is:

$$\begin{aligned} \max \quad & z = CX + 0X_S \\ s.t. \quad & AX + IX_S = b \\ & X, X_S \ge 0 \end{aligned} \tag{2}$$

For a chosen feasible basis B, we may assume that $B = (P_1, P_2, \ldots, P_m)$, $N = (P_{m+1}, P_{m+2}, \ldots, P_n)$ is the coefficient matrix of the non-basic variables, which is denoted as $A = (B, N)$, where $P_j (j = 1,2,3,\ldots,m)$ represents the j th column of matrix A. Correspondingly, we denote

$$X = \begin{pmatrix} X_B \\ X_N \end{pmatrix}, C = (C_B, C_N)$$

The dual model of Model (1) is

$$\begin{aligned} \min \quad & \omega = Yb \\ s.t. \quad & YA \ge C \\ & Y \ge 0 \end{aligned} \tag{3}$$

Its standard form is

$$\begin{aligned} \min \quad & \omega = Yb + 0Y_S \\ s.t. \quad & YA - Y_S I = C \\ & Y, Y_S \ge 0 \end{aligned} \tag{4}$$

3. Teaching Design for the Dual Simplex Method

In teaching the dual simplex method, the primary objective is to help students understand the essence of solving linear programming problems using the dual simplex method and to master the computational steps involved. The teaching focus lies on the computational procedures of the dual simplex method, while the main difficulty is distinguishing the differences between the principles of solving linear programming via the simplex method and the solution approach of the dual simplex method compared to the primal simplex method.

Through a careful study of dual simplex method textbooks, we found that most textbooks describe the relationship between the primal and dual problems starting from Form (1) and then derive the matrix forms. This approach is suitable for students with strong mathematical foundations; however, for students with weaker mathematical backgrounds, the content often appears abstract and difficult to grasp. Therefore, this paper proposes a teaching plan that provides a clear and accessible explanation of the relationship between the primal and dual problems, as well as the principles of the dual simplex method.

ISSN: 2637-6067 DOI: 10.6918/IJOSSER.202510_8(10).0006

3.1. Presenting the Matrix Description of the Simplex Method Through a Practical Example

Consider the following linear programming model:

$$\max z = 3x_1 + 4x_2 + x_3$$
$$s.t.\begin{cases} 6x_1 + 5x_2 + 3x_3 \le 45 \\ 3x_1 + 5x_2 + 4x_3 \le 30 \\ x_1, x_2, x_3 \ge 0 \end{cases} \quad (5)$$

After converting it into the standard form according to Form (2), an obvious feasible basis can be identified, as the basis of the basic variables forms a clearly feasible basis. The corresponding simplex tableau for the above LP standard form is as follows:

Table 1. Initial Simplex Tableau of the Example

C_B	Basis	b	x_1	x_2	x_3	x_4	x_5
0	x_4	45	6	5	3	1	0
0	x_5	30	3	5	4	0	1
σ_j			-3	-4	-1	0	0

Through a series of iterations, the following optimal simplex tableau can be obtained:

Table 2. Optimal Simplex Tableau of the Example

C_B	Basis	b	x_1	x_2	x_3	x_4	x_5
3	x_1	5	1	0	-1/3	1/3	-1/3
4	x_2	3	0	1	1	-1/5	2/5
σ_j			0	0	-2	-1/5	-3/5

Based on the above example, the matrix form of the simplex method \[1] can be summarized. The initial simplex tableau for Form (2) is shown in Table 3:

Table 3. Matrix Form of the Initial Simplex Tableau

		Non-basic variables		basic variables	
C_B	Basis	b	X_B	X_N	X_s
0	X_s	b	B	N	I
σ_j			C_B	C_N	0

Table 1 leads to the optimal simplex tableau through the simplex method, as shown in

Table 4. Matrix Form of the Optimal Simplex Tableau

		Non-basic variables		basic variables	
C_B	Basis	b	X_B	X_N	X_s
C_B	X_B	B-1b	I	B-1N	B-1
σ_j			0	$C_N - C_B B^{-1}$	$-C_B B^{-1}$

Introducing the matrix form of the simplex method through a specific example helps students intuitively understand the matrix transformations from the initial simplex tableau to the optimal simplex tableau, facilitating their mastery of sensitivity analysis and related computations.

Based on this, the theoretical derivation of the relationship between the solutions of the primal and dual problems can be presented. When *B* is the optimal basis, all the reduced costs should be less than zero ($\sigma_j \le 0$), which, for Table 4, corresponds to:

$$C_N - C_B B^{-1} N \le 0, -C_B B^{-1} \le 0$$

And the test number for X_B can be written as: $C_N - C_B B^{-1} B \le 0$. So formulas (1), (2), and (3) can be re-expressed as:

$$C - C_B B^{-1} A \le 0, -C_B B^{-1} \le 0$$

If we let $Y = C_B B^{-1}$, then the above formula can be expressed as:

$$YA \ge C, \quad Y \ge 0$$

It satisfies the dual problem (3). At this point, it can be observed that taking the opposite of the test number row yields exactly a feasible solution to the dual problem. Substituting this solution into the objective function of the dual problem,

$$\omega = Yb = C_B B^{-1} b = Z$$

It can be observed that when the primal problem attains its optimal solution, its dual problem has a feasible solution, and the two problems share the same objective function value. Later, in the discussion of the properties of dual problems, it will be shown that the solution to the dual problem is also its optimal solution under this condition.
The following uses specific examples to illustrate the implications of the above theoretical derivation.

3.2. Relationships Between the Solutions of the Dual Problem and the Primal Problem

Using the same example (5) as above, we write its dual problem as follows:

$$\min \omega = 45y_1 + 30y_2$$

$$s.t.\begin{cases} 6y_1 + 3y_2 \ge 3 \\ 5y_1 + 5y_2 \ge 4 \\ 3y_1 + 4y_2 \ge 1 \\ y_1, y_2 \ge 0 \end{cases}$$

By solving the dual problem, we can obtain its optimal simplex tableau.

ISSN: 2637-6067 DOI: 10.6918/IJOSSER.202510_8(10).0006

Table 5. Optimal Simplex Tableau for the Dual Problem of the Illustrative Example

C_B	Basis	b	y_1	y_2	y_3	y_4	y_5
45	y_1	1/5	1	0	1/3	-1	0
30	y_2	3/5	0	1	-1/3	1/5	0
0	y_5	2	0	0	1/3	-2/5	1
σ_j			0	0	5	3	0

Relationships between the variables of the two problems can be observed from Table 2 and Table 5. In teaching, emphasis should be placed on three key types of relationships to facilitate the explanation of the principle of the dual simplex method: the relationship between the optimal solution of the primal problem and the test numbers of the dual problem, the relationship between the optimal solution of the dual problem and the test numbers of the primal problem, and the relationship between their coefficients. The main relationships are as follows:

(1) The slack variables of the primal problem correspond to the variables of the dual problem, and the variables of the primal problem correspond to the surplus variables of the dual problem;

(2) The optimal solution of the dual problem is equal to the negative of the test numbers in the optimal simplex tableau of the primal problem, and the optimal solution of the primal problem is equal to the test numbers in the optimal simplex tableau of the dual problem;

(3) The coefficient matrix of non-basic variables in the optimal simplex tableau of the dual problem is equal to the negative of the transpose of the coefficient matrix of non-basic variables in the optimal simplex tableau of the primal problem.

3.3. Solution Principle of the Dual Simplex Method

After comparing the two optimal simplex tableaus above, the solution principle of the dual simplex method is explained based on the above analysis, which specifically includes the following:

(1) The solution principle of the simplex method is proposed: under the condition that the basic feasible solution of the primal problem is maintained to be greater than 0, the basis matrix B is continuously updated through iterations, so that the test numbers reach the optimal state.

(2) Clarify the relationship between the solution process of the simplex method for the dual problem and that for the primal problem. In the iteration process of the simplex method, the basic feasible solution is maintained to be greater than or equal to 0, and the test numbers are continuously optimized until they finally reach the optimal state. However, in the process of solving the dual problem using the simplex method, maintaining the basic feasible solution of the dual problem to be greater than 0 is equivalent to maintaining the optimality of the test numbers in the primal problem; meanwhile, optimizing the test numbers of the dual problem is equivalent to optimizing the b-column in the primal problem, until the b-column finally becomes a basic feasible solution.

(3) Clarify the principle behind the rule for selecting the pivot entry ratio. In the process of solving the primal problem using the simplex method, the entering variable x_k is determined based on $\max(\sigma_j) = \sigma_k$; when determining the departing variable, calculations are performed in accordance with the relevant rule.

$$\theta = \min\left(\frac{b_i}{a_{ik}} \mid a_{ik} > 0\right) = \frac{b_l}{a_{lk}}$$

ISSN: 2637-6067 DOI: 10.6918/IJOSSER.202510_8(10).0006

By comparing the relationship between the optimal simplex tableau of the primal problem and that of the dual problem, this ratio transforms into the ratio of test numbers to coefficients in the tableau when applying the dual simplex method. In the dual simplex method for maximization problems, the test numbers are always less than or equal to 0. Since the coefficient matrix of non-basic variables (in the dual problem) is the negative of the transpose of the coefficient matrix of non-basic variables in the primal problem, the denominator is required to satisfy specific conditions when calculating the ratio—ensuring the ratio is always non-negative. In this case, the smallest ratio is still selected (as in the primal simplex method). For minimization problems, the test numbers are always greater than or equal to 0. Due to the characteristics of the denominator (consistent with the dual relationship), the smallest value among the absolute values of the ratios must be selected at this stage.

After explaining the principle of the dual simplex method, the solution steps for the method are provided below. Additionally, the advantages of the dual simplex method are highlighted: for linear programming problems with redundant variables or constraints, it can reduce computational effort and simplify operations by eliminating the need to introduce artificial variables.

3.4. Differences and Connections Between the Dual Simplex Method and the Simplex Method

The simplex method is the primary approach for solving linear programming problems. The dual simplex method applies the simplex method to the computation of dual problems, which can enhance the efficiency of solving linear programming problems. The differences and connections between the two methods are as follows:

(1) Different underlying principles: The simplex method iterates toward the feasibility of the dual problem on the basis of ensuring the primal problem has a feasible solution; in contrast, the dual simplex method iterates toward the feasibility of the primal problem under the condition of ensuring the dual problem has a feasible solution.

(2) Different pivot selection sequences: For the simplex method, the entering variable is determined first, followed by the departing variable; whereas for the dual simplex method, the departing variable is determined first, and then the entering variable.

(3) Different iteration rules: The simplex method selects entering and departing variables in accordance with the "maximum-minimum" principle; the dual simplex method, however, follows the "minimum-minimum" principle for selecting departing and entering variables.

(4) Different optimality conditions: For the simplex method, optimality is determined by checking whether all test numbers of non-basic variables are less than or equal to 0; for the dual simplex method, optimality is determined by checking whether all elements in$B^{-1}b$ (in the dual simplex tableau) are greater than or equal to 0.

(5) Shared nature as solution methods: Both the simplex method and the dual simplex method are techniques for solving linear programming problems; the only difference is that the dual simplex method is implemented from the perspective of the dual problem.

4. Summary of Teaching Methods

In the teaching of Operations Research, knowledge related to the dual simplex method has always been a difficult point for students to understand and master. During the learning process, students often struggle to comprehend the relationships between the solutions of the primal and dual problems, as well as the underlying principles of the dual simplex method. Through the design of the teaching strategy for the dual simplex method proposed in this paper, the principles are explained systematically, clearly, and progressively using practical examples, making complex concepts accessible. The teaching strategy has the following characteristics:

(1) Matrix representation of the simplex method starts with practical examples: The matrix description of the simplex method is introduced using actual examples, followed by an analysis of the matrix evolution process. This approach helps students understand the entire transformation process intuitively, laying a solid foundation for deriving the relationships between the optimal solutions of the primal and dual problems.

(2) Connecting matrix form to theoretical and practical verification: Building on the matrix form of the simplex method, the paper first explains the theoretical relationship between the test numbers in the optimal simplex tableau and the optimal solution of the dual problem. It then uses examples to demonstrate this relationship visually, and further summarizes the connections between the optimal simplex tableaus of the primal and dual problems—preparing students for the subsequent explanation of the solution principle of the dual simplex method.

(3) Deriving principles through comparison and practical perception: Based on the connections between the optimal simplex tableaus of the primal and dual problems, the solution principle of the dual simplex method is proposed by comparing it with the principle of the simplex method, and the corresponding solution steps are provided. After in-depth example analysis, students can develop an intuitive understanding of the connections between the two optimal simplex tableaus, leading to a more thorough grasp of the essence of the dual simplex method.

(4) Summarizing differences and connections for comparative application: Finally, the differences and connections between the simplex method and the dual simplex method are summarized. This facilitates students' comparative understanding of the two methods and enhances their ability to apply them flexibly.

Acknowledgements

Natural Science Foundation.

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